# A Brief Introduction to Finite Element methods for flow problems

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# Outline

# Motivation: Pressure in Navier-Stokes

## Incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}, \qquad \text{(momentum)}$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \text{(continuity)}$$

Taking divergence

$$\underbrace{\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{u}\right)}_{=0} + \nabla\cdot\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla^2 p + \underbrace{\nu\nabla^2\nabla\cdot\boldsymbol{u}}_{=0}.$$

I.e.

$$\nabla^2 p = -\nabla\cdot\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right).$$

# Poisson Equation

General form

**Poisson Equation**

$$\nabla^2 \psi = f(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega \qquad (*)$$

In 1D, setting $\Omega$ to the unit interval:

$$\frac{\partial^2 \psi}{\partial x^2} = f(x) \quad \forall x \in (0,1).$$

This is the strong form of the Poisson equation.

# Strong Form vs. Weak Form of an Equation

## Strong form

Equation (*) true individually for each point in space,

$$\nabla^2\psi = f(x)$$

for all $x$ in domain $\Omega$.
Test a $\psi$ by checking equation holds individually for each point in space.

## Weak form

Integral equation holds for all choices of a 'test function', $\phi$,

$$\int_\Omega \phi\left(\nabla^2\psi - f\right)\,dV = 0,$$

where $\phi$ is any function $\phi : \Omega \to \mathbb{R}$ from a function space to be defined later.
Test a $\psi$ ('trial function') by checking integral equation holds for all test functions,$\phi$.

# Strong and weak forms

A solution to the strong form of the equations will be a solution to the weak form equations:
If $\nabla^2\psi = f$ then

$$\int \phi \left(\nabla^2\psi - f\right) \, dV = \int \phi \cdot 0 \, dV = 0,$$

independent of $\phi$, i.e. for any possible choice of test space.

# Strong and weak forms

A solution to the weak form of the equations may be a solution to the strong equations if it is smooth enough.
The weak formulation extends the equations to allow non-smooth solutions which exist in a distributional sense.

## Examples of common distributions

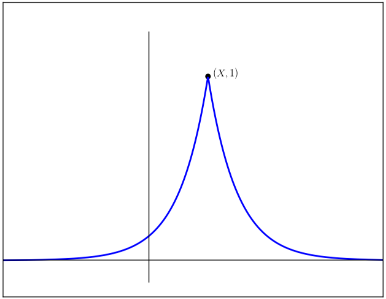
$$\delta(x), \quad \int_{-\infty}^{a} f(x)\,\delta(x) = \begin{cases} f(0) & a > 0, \\ 0 & a < 0. \end{cases} \qquad \text{(Dirac delta)}$$

$$H(x) = \int_{-\infty}^{x} \delta(s)\,ds = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases} \qquad \text{(Heaviside)}$$

# Example of a weak non-classical solution

$$\psi - \nabla^2\psi(x) = a\delta(X - x) \qquad \text{(Helmholtz)}$$

$$\psi = \begin{cases} a\exp(-x + X) & x \leq X, \\ a\exp(x - X) & x > X \end{cases}$$



# Boundary Conditions

Two possible forms of boundary condition for a solution to the Poisson equation to be well posed:

1. Dirichlet: $\psi(x) = a(x)$ for $x \in A \subset \delta\Omega$,
2. Neumann: $\frac{\partial \psi}{\partial x} = b\left(x\right)$ for $x \in B \subset \delta\Omega$.

In Galerkin formulation Dirichlet boundary conditions require explicit modification of the problem solves, whereas Neumann conditions are dealt with naturally as part of the formulation. We'll use a Dirichlet boundary condition at $x = 0$, and a Neumann condition at $x = 1$here.

# Natural Boundary Conditions

$$\int_0^1 \phi \left( \frac{\partial^2 \psi}{\partial x^2} - f \right) dx = 0,$$

Integrate equation by parts,

$$\int_0^1 \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} dx = - \int_0^1 \phi f dx + \left[ \phi \frac{\partial \psi}{\partial x} \right]_0^1 .$$

Chose $\phi$ to vanish on Dirichlet boundaries (and set $\psi = a$), and use our knowlege of $\frac{\partial \psi}{\partial x}$ on Neumann boundaries:

$$\int_0^1 \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} dx = \int_0^1 \phi f \, dx + \phi\left(1\right) b\left(1\right)$$

# Finite Element Functions

Need a discrete finite dimensional representation of problem to do numerical calculations on a computer. Set

$$\psi^{\delta}(x) = \sum_{i=1}^{N} \hat{\psi}_i N_i(x)$$

where $\hat{\psi}_i \in \mathbb{R}$ is a scalar parameter and $N_i : \Omega \to \mathbb{R}$ is a fixed shape function.

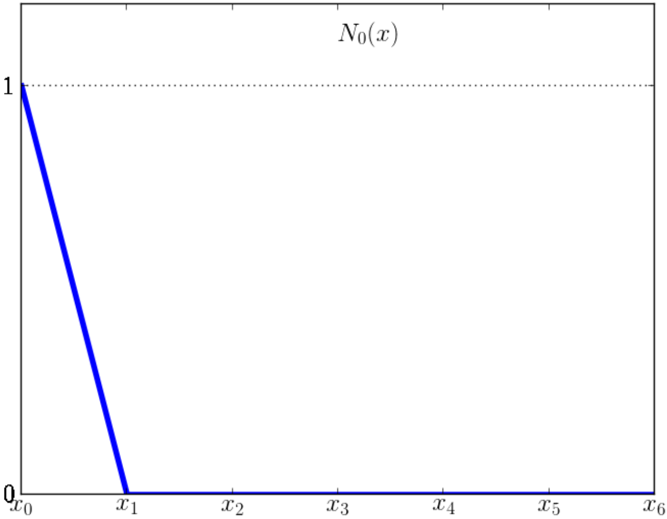
# Finite Element Functions

For our 1D Poisson equation example we can choose to use the set of continuous, piecewise linear functions ('shape functions')on subdivisions of the unit interval.

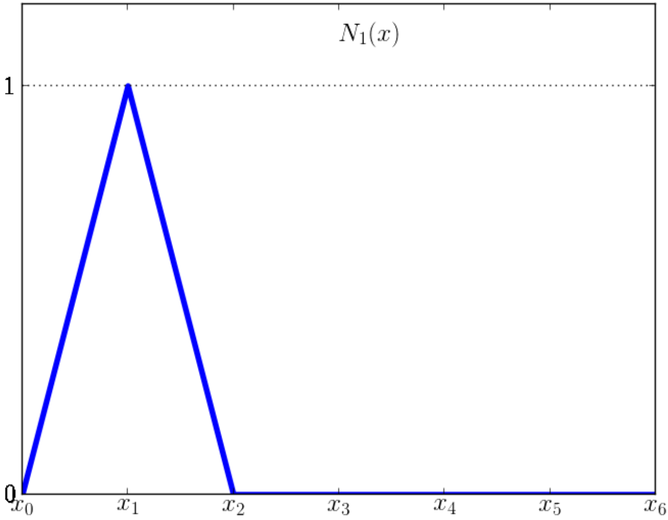
$$N_i = \begin{cases} 0, & x \leq x_{i-1}, \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}, & x_{i-1} < x \leq x_i, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & x_I < x \leq x_{i+1}, \\ 0. & x > x_{i+1}. \end{cases}$$

Functions are equal to 1 at the set of points $[0, x_1, x_2, \ldots x_{n-1}, 1]$.
The subdivisions of $\Omega$ are often called elements.

# Finite Element Functions



# Finite Element Functions



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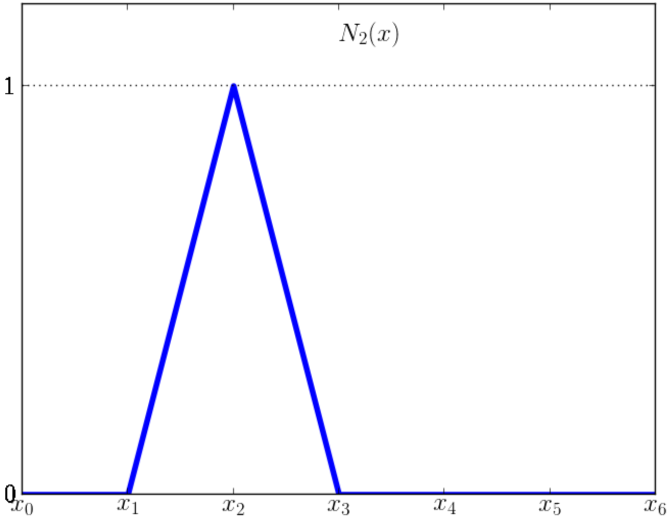

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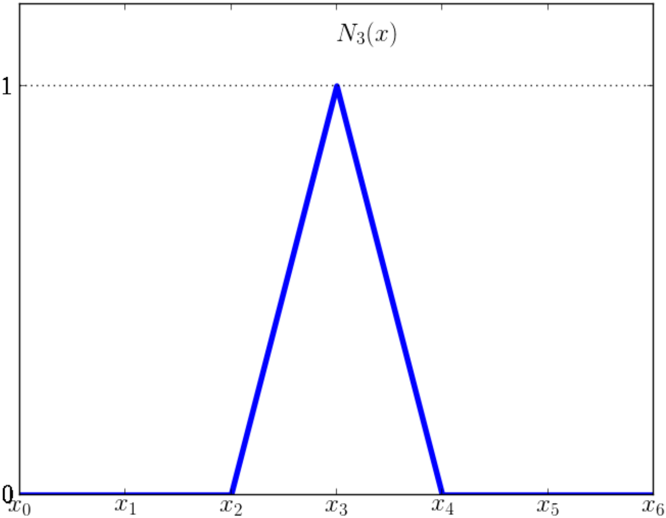

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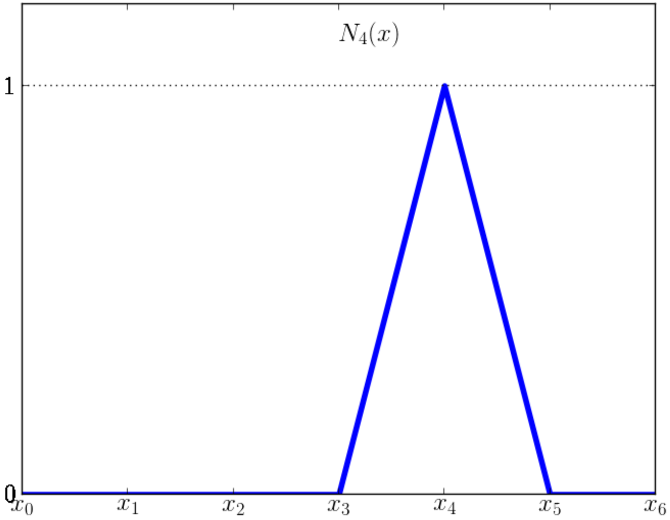

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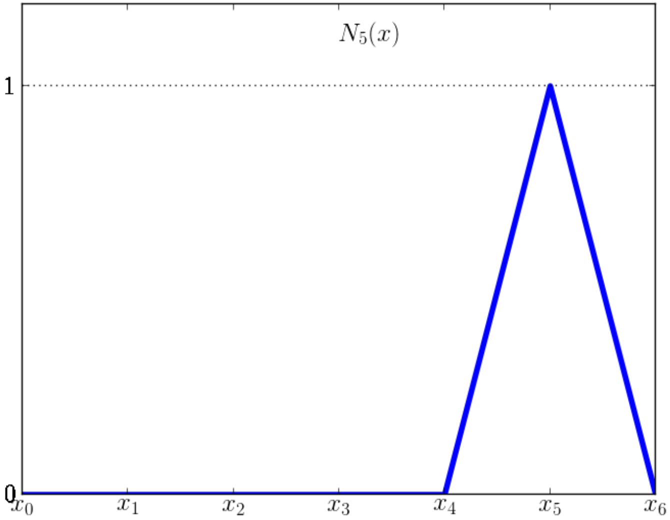

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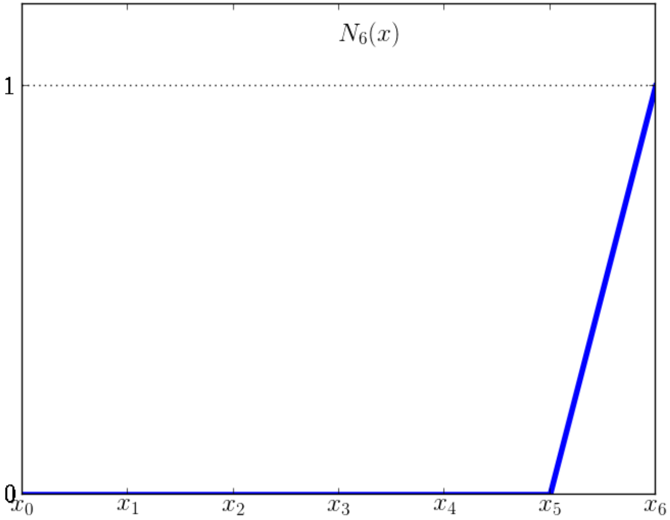


# Galerkin Approximation

To obtain the Galerkin approximation of the Poisson equation, we find the solution of the weak form when $\psi$ and $\phi$ are approximated by our finite element expansions,

$$\psi^\delta = \sum_{i=0}^{n} \hat{\psi}_i N_i,$$

$$\phi^\delta = \sum_{j=0}^{n} \hat{\phi}_j N_j.$$

The $\psi^\delta$ are called trial functions and the function space they come from is the trial space. The $\phi^\delta$are called test functions and live in the trial space. Computation involves obtaining the finite number of $\hat{\psi}_i$. Can be solved on a computer.

# Galerkin Approximation

Substituing the finite representations into (*) we get

$$\int \sum_{j=1}^{n} \hat{\phi}_j \frac{\partial N_j}{\partial x} \sum_{i=1}^{N} \hat{\psi}_i \frac{\partial N_i}{\partial x}\, dV - v_N^{\delta} b\left(x_N\right) = \int \sum_{j=1}^{n} \hat{\phi}_j^{\delta} N_j f\, dV,$$

$$\hat{\phi}_j \left( \left[ \int \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x}\, dV \right] \hat{\psi}_i - \int f N_j\, dV \right) = \begin{cases} 0, & j = 1, \ldots n-1 \\ \hat{\phi}_n b\left(x_n\right), & j = n \end{cases}$$

if term in brackets vanishes, solution have a solution for any value of $\hat{\phi}_j$, so we can ignore them.

# The Right Hand Side

In general $f$ is known explicitly as a function $f : \Omega \to \mathbb{R}$. Hence the integral $\int_0^1 \phi^\delta f \, dx$ can theoretically be calculated exactly. In actual practice (especially for coupled problems) it is usually represented in the approximate function space,

$$f^\delta(x) = \sum_{i=1}^{N} \hat{f}_i N_i(x),$$

where, for our choice of shape functions,

$$\hat{f}_i = f(x_i).$$

# Matrix problem

Dirichlet condition: $\hat{\psi}_0 = a(0)$, turns up on right hand side

$$
\begin{bmatrix}
-\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 & 0 \\
\frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 & 0 \\
0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 & 0 \\
0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & 0 \\
0 & 0 & 0 & \frac{1}{h} & -\frac{2}{h} & \frac{1}{h} \\
0 & 0 & 0 & 0 & \frac{1}{h} & -\frac{1}{h}
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6
\end{bmatrix}
=
\begin{bmatrix}
\frac{h}{6}\hat{f}_0 + \frac{2h}{3}\hat{f}_1 + \frac{h}{6}\hat{f}_2 - \frac{a(0)}{h} \\
\frac{h}{6}\hat{f}_1 + \frac{2h}{3}\hat{f}_2 + \frac{h}{6}\hat{f}_3 \\
\frac{h}{6}\hat{f}_2 + \frac{2h}{3}\hat{f}_3 + \frac{h}{6}\hat{f}_4 \\
\frac{h}{6}\hat{f}_3 + \frac{2h}{3}\hat{f}_4 + \frac{h}{6}\hat{f}_5 \\
\frac{h}{6}\hat{f}_4 + \frac{2h}{3}\hat{f}_5 + \frac{h}{6}\hat{f}_6 \\
\frac{h}{6}\hat{f}_5 + \frac{h}{3}\hat{f}_6 \text{-b(1)}
\end{bmatrix}
$$

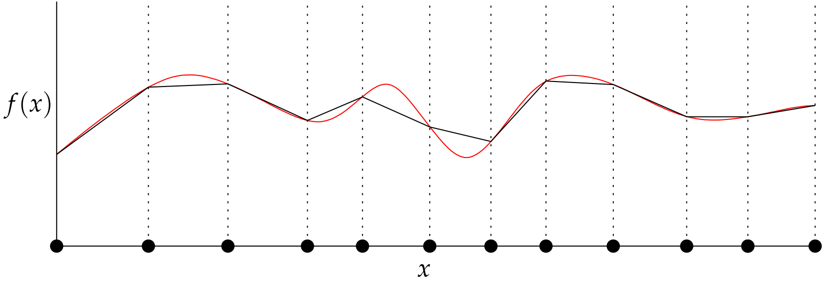
# Increasing the degrees of freedom

To increase the number of free parameters in the approximate solution space (and thus attempt to get a more accurate solution) there are several options:

- More, smaller subdivisions [step size,$h$]
    - This is the system used in Fluidity's mesh adaptivity.
- Use higher order polynomials, e.g. quadratic functions [polynomial order, $p$]
- Use discontinuous functions [Discontinuous Galerkin formulation]

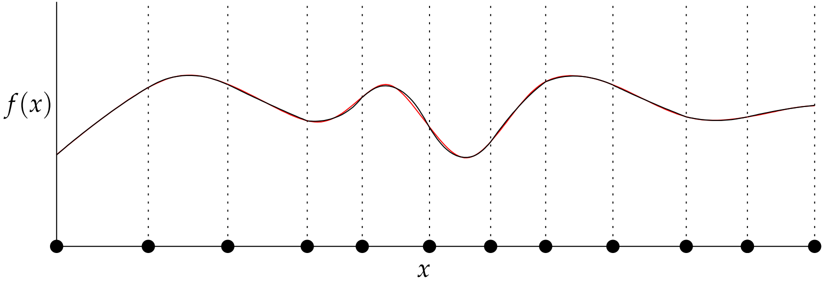
# Increasing the degrees of freedom

- Projection of a function.
- Linear, continuous shape functions, Galerkin method (P1 CG).
- In 1d degrees of freedom ≈no. of elements.



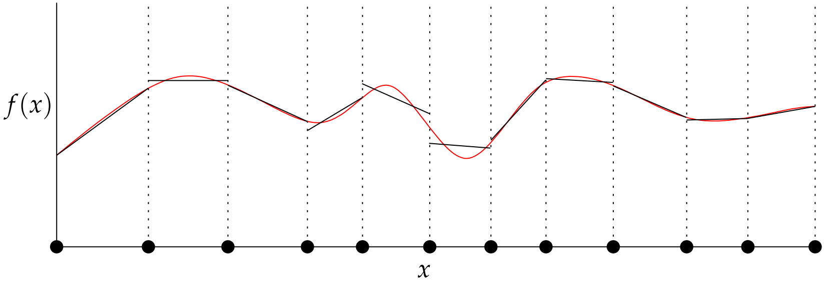
# Increasing the degrees of freedom

- Projection of a function.
- Quadratic, continuous shape functions, Galerkin method (P2 CG).
- In 1d degrees of freedom $\approx 2 \times$ no. of elements.
- Good for smooth functions



# Increasing the degrees of freedom

- Projection of a function.
- Linear, discontinuous shape functions, Galerkin method (P1 DG).
- In 1d degrees of freedom ≈no. of elements.
- Good for discontinuties/large gradients.



# Review of Section

- Finite element methods solve weak (integral) equations
- Functions get approximated by finite dimensional summations of functions, which are non-zero on small regions of the problem domain (elements)
- Linear PDE problem gives a linear (matrix) problem for the $\hat{\psi}_i$.

The efficient computational representation and solution of these sorts of problems will form the other two sessions in the day (see lectures by David Ham and Stephan Kramer).

# Section II: Reassuring Maths

This section summarises some useful results from mathematical analysis for finite element problems. In particular, we note that results exists to show that, under certain provisos finite element solutions to a given problem

- exist
- are unique
- converge
- converge to the right answer.

# Vector Spaces

Going back to the weak form for the original infinite dimensional problem,

$$\int_\Omega \nabla\phi\cdot\nabla\psi\,dV = \int_\Omega \phi f\,dV + \int_{\delta\Omega^N}\phi b\,dV,$$

it is obvious that $\psi$ and $\nabla\psi$ must be well behaved enough for these integrals to exist.

# Vector Spaces

We require the function is square integrable,

$$\|\psi\|^2 := \int_\Omega \psi^2 \, dV < \infty. \tag{1}$$

(The space of function which satisfy this is normally called $\mathcal{L}_2(\Omega)$) and also that

$$\|\nabla\psi\|^2 := \int_\Omega \nabla\psi \cdot \nabla\psi \, dV < \infty, \tag{2}$$

Functions which satisfy (1) & (2) are in the space of square integrable functions with square integrable derivatives, denoted $\mathcal{H}^1(\Omega)$. This is a Sobelov space.

# Quick Review of Vector Spaces

A set, $\mathcal{V}$, is a vector space if is has addition and scalar multiplication operators where

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c},$$
$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a},$$
$$\text{there exists } \mathbf{0} \text{ such that } \boldsymbol{a} + \mathbf{0} = \boldsymbol{a}, \text{ for all } \boldsymbol{a} \in \mathcal{V},$$
$$\text{for all } \boldsymbol{a} \text{ there exists } -\boldsymbol{a}\text{such that } \boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0},$$
$$\alpha\,(\boldsymbol{a} + \boldsymbol{b}) = \alpha\boldsymbol{a} + \alpha\boldsymbol{b},$$
$$(\alpha + \beta)\,\boldsymbol{a} = \alpha\boldsymbol{a} + \beta\boldsymbol{a},$$
$$\alpha\,(\beta\boldsymbol{a}) = (\alpha\beta)\,\boldsymbol{a},$$
$$1\boldsymbol{a} = \boldsymbol{a}.$$

# Vector spaces

Functions are a vector space under the following definitions of addition and scalar multiplication:

$$(f+g)(x) = f(x) + g(x),$$
$$(\alpha f)(x) = \alpha(f(x)).$$

I.e. functions are added/multiplied pointwise based on their result. Conditions all follow from the normal rules of addition/multiplication.

# Weak equations and Bilinear forms

The volume integral in (*) defines a symmetric bilinear form, $a : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$,

$$a(\phi,\psi) := \int_\Omega \nabla\psi \cdot \nabla\phi \, d^n x.$$

where

$$\begin{aligned} a(\phi,\psi) &= a(\psi,\phi), \\ a(c_1\phi + c_2\xi, \psi) &= c_1 a(\phi,\psi) + c_2 a(\xi,\psi). \end{aligned}$$

# Lax-Milgram Theorem

Two properties of the bilinear form, $a$, are used to show well-posedness:

$$a(\phi,\psi) \leq C \left\|\psi\right\| \left\|\phi\right\| \text{ for some C} > 0, \qquad \text{(continuity)}$$

$$a(\psi,\psi) \geq c \left\|u\right\|^2 \text{ for some c>0.} \qquad \text{(coercive/elliptic)}$$

# Well posedness - uniqueness:

Suppose there are two different solutions, $\psi_1$ and $\psi_2$, i.e.

$$a(\phi, \psi_1) = a(\phi, \psi_2) = \int_\Omega \phi f \, d^n x, \text{ for all } \phi \in \mathcal{H}^1(\Omega).$$

Then

$$a(\phi, \psi_1 - \psi_2) = 0$$

but $\psi_1 - \psi_2 \in \mathcal{H}^1(\Omega)$. Then

$$a(\psi_1 - \psi_2, \psi_1 - \psi_2) = 0 \geq c \|\psi_1 - \psi_2\|^2,$$

So $\psi_1 = \psi_2$, hence solution is unique.

# Summary

- Finite element methods solve a weak form of the exact equations in an approximate solution space.
- The approximate solution is defined (almost) everywhere.
- Neuman conditions dealt with implicitly inside formulation
- Dirichlet conditions (as in finite difference methods) appear in right hand side.

# For Further Reading

J. Donea & A. Huerta
Finite Elements Methods for Flow Problems.
Wiley 2003